Stabilization in Distribution of Hybrid Systems by Intermittent Noise

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Outline

1. Background
2. Main results
3. An example

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- Given an unstable hybrid system

$$dx(t)/dt = f(x(t), r(t)), \tag{1.1}$$

it is required to find a feebdback control $\sigma(x(t), r(t))dw(t)$, so that the controlled system

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Stabilization by random noise has been studied intensively by many authors. $(r(t) = 0)$

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Given an unstable nonlinear system (1.1) with $r(t) = 0$, Zhang et al designed a feedback control $g(x(t))\beta(t)dw(t)$, in the diffusion part so that the corresponding stochastic system

$$dx(t) = f(x(t))dt + g(x(t))\beta(t)dw(t) \tag{1.3}$$

was almost surely exponentially stable. Here $\beta : [0, \infty) \to \{0, 1\}$ is defined by

$$\beta(t) = \sum_{n=0}^{\infty} I_{[nT, nT+\theta T)}(t), t \geq 0,$$

where $T > 0$ denotes the control period and $\theta T > 0$ is the working width satisfying $\theta \in (0, 1)$.

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Most of these papers are concerned with asymptotic stability in probability or in mean square (i.e. the solution will tend to zero in probability or in mean square).

However, this asymptotic stability is sometimes too strong. For example, for many population systems under random environment, the stochastic permanence is a more desired control objective than the extinction.

In this situation it is useful to investigate whether or not the probability distribution of the solutions will converge to a probability distribution, but not to zero. This property is called asymptotic stability in distribution.

The stability in distribution is to study if the probability distributions of the solutions of a hybrid system will converge to a probability distribution, known as stationary distribution.

- G. K. Basak, R. N. Bhattacharya, Ann. Probab, (1992)
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Assume that the given hybrid ODE is not stable in distribution. The first problem we are going to investigate in this talk is:

- **Problem 1**: Is it possible to design a feedback control using Brownian noise to make the stochastically controlled SDE

$$dX(t) = f(X(t), r(t))dt + u(X(t), r(t))dB(t) \tag{1.4}$$

to become stable in distribution?

- **Problem 2**: Is it possible to design an intermittent stochastic feedback control to make the stochastically controlled SDE

$$dX(t) = f(X(t), r(t))dt + \beta(t)u(X(t), r(t))dB(t) \tag{1.5}$$

to become stable in distribution?

Here $\beta : [0, \infty) \to \{0, 1\}$ is defined by

$$\beta(t) = \sum_{k=0}^{\infty} I_{[kh,(k+1-\delta)h)}(t), \quad t \geq 0, \tag{1.6}$$

where $h > 0$, $\delta \in [0, 1)$ and $I_{[kh,(k+1-\delta)h)}(t)$ is the indicator function of $[kh, (k+1-\delta)h)$, namely it takes 1 when $t \in [kh, (k+1-\delta)h)$ and 0 otherwise.

Remark 1

The parameter δ is the proportion of rest in one period of h or in long term. In the case when $\delta = 0$, $\beta(t) = 1$ for all $t \geq 0$ so the stochastic control acts without any rest, and Problem 2 becomes Problem 1.

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions
- Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space.
- Let $r(t)$, $t \geq 0$, be a right-continuous irreducible Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

  where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$.

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Problem 2

Assumption 2

There are constants $a_i \geq 0$, $b_i \geq 0$ and $c_i \geq 0$ ($i \in S$) such that

$$\begin{aligned} (x-y)^T(f(x,i)-f(y,i)) &\leq a_i|x-y|^2, \\ |u(x,i)-u(y,i)| &\leq b_i|x-y|, \\ |(x-y)^T(u(x,i)-u(y,i))| &\geq c_i|x-y|^2, \end{aligned}$$

for all $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$.

Problem 2

- It is known that the joint process $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t))$ is a Markov process on $t \geq 0$. Due to the intermittent term $\beta(t)$, it is not time-homogeneous.
- Fortunately, $\beta(t)$ is a periodic function with its period h. For example, we observe that $\{(X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in N_+}$ forms a discrete-time $\mathbb{R}^n \times S$-valued time-homogeneous Markov process.
- Denote by $P(k, \hat{x}, \hat{i}; dy \times \{j\})$ its k-step transition probability measure, namely

$$P(k, \hat{x}, \hat{i}; A \times B) = \mathbb{P}((X_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh)) \in A \times B)$$

for any $A \in \mathcal{B}(\mathbb{R}^n)$ and $B \subset S$.

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- Denote $\mathcal{C}_h$ the family of continuous functions ξ from $[0, h]$ to $\mathbb{R}^n$ with norm $\|\xi\|_h = \sup_{s\in[0,h]} |\xi(s)|$.
- Denote by $\mathcal{P}(\mathcal{C}_h)$ the family of probability measures on $\mathcal{C}_h$. For $P_1, P_2 \in \mathcal{P}(\mathcal{C}_h)$, define metric d_Φ by

$$d_\Phi(P_1, P_2) = \sup_{\phi\in\Phi} \left| \int_{\mathcal{C}_h} \phi(\xi) P_1(d\xi) - \int_{\mathcal{C}_h} \phi(\xi) P_2(d\xi) \right|$$

where

$$\Phi = \{\phi : \mathcal{C}_h \to \mathbb{R} \text{ satisfying } |\phi(\xi) - \phi(\zeta)| \le \|\xi - \zeta\|_h \text{ and } |\phi(\xi)| \le 1 \text{ for } \xi, \zeta \in \mathcal{C}_h\}.$$

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$$\begin{aligned}\Phi =& \{\phi : \mathcal{C}_h \to \mathbb{R} \text{ satisfying } |\phi(\xi) - \phi(\zeta)| \le \|\xi - \zeta\|_h \\ & \text{and } |\phi(\xi)| \le 1 \text{ for } \xi, \zeta \in \mathcal{C}_h\}.\end{aligned}$$

Problem 2

- For $k \in N_+$, define $\tilde{X}_{\hat{x},\hat{i}}(kh) = \{X_{\hat{x},\hat{i}}(kh+s) : 0 \le s \le h\}$ which is $\mathcal{C}_h$-valued. Then $\{(\tilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))\}_{k\in N_+}$ forms a discrete-time $\mathcal{C}_h \times S$-valued time-homogeneous Markov process.
- In fact, the time-homogeneous property follows from the periodic property of $\beta(\cdot)$.
- To see why this process is Markov, we observe that once once $\{(\tilde{X}_{\hat{x},\hat{i}}(k_1 h), r_{\hat{i}}(k_1 h))\}_{k_1\in N_+}$ for some $k_1 \in N_+$ is given, $(X_{\hat{x},\hat{i}}(k_1 h), r_{\hat{i}}(k_1 h))$ is known and then $(X_{\hat{x},\hat{i}}(t), r_{\hat{i}}(t))$ for all $t \ge k_1 h$, namely $(\tilde{X}_{\hat{x},\hat{i}}(kh), r_{\hat{i}}(kh))$ for all $k \ge k_1$, can be uniquely determined by solving the SDE (1.5) with initial data $(X_{\hat{x},\hat{i}}(k_1 h), r_{\hat{i}}(k_1 h))$ at time $k_1 h$.

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Problem 2

Definition 3

The controlled SDE (1.5) is said to be asymptotically stable in distribution if there exists a probability measure $\mu_h \in \mathcal{P}(\mathcal{C}_h)$ such that

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\tilde{X}_{\hat{x},\hat{i}}(kh)), \mu_h) = 0$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times S$.

Problem 2

Assumption 4

There is a constant $p \in (0,1)$ such that

$$\mathcal{A} := \mathrm{diag}(\zeta_1 - pa_1, \cdots, \zeta_N - pa_N) - \Gamma \tag{2.1}$$

is a nonsingular M-matrix, where

$$\zeta_i = 0.5p[(2-p)c_i^2 - b_i^2], \ \ i \in S \tag{2.2}$$

and a_i, b_i, c_i are the same in Assumption 2.

Problem 2

- Define

$$(\theta_1, \cdots, \theta_N)^T = \mathcal{A}^{-1}(1, \cdots, 1)^T,$$

by the theory of M-matrices, $\theta_i > 0$ for all $i \in S$.

- Set

$$\underline{\theta} = \min_{1 \le i \le N} \theta_i, \ \bar{\theta} = \max_{1 \le i \le N} \theta_i, \ \sigma = \max_{1 \le i \le N} \zeta_i.$$

and

$$\delta^* = 1 \wedge (1/(\sigma\bar{\theta})), \ (\sigma > 0).$$

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Problem 2

Lemma 5

Under Assumption 2,

$$\mathbb{P}(X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t) \neq 0 \text{ for all } t \geq 0) = 1 \qquad (2.3)$$

for any $\hat{x}, \hat{y} \in \mathbb{R}^n$ with $\hat{x} \neq \hat{y}$ and $\hat{i} \in S$. That is, two solutions starting from two different states but the same mode will never meet each other with probability one.

Problem 2

Lemma 6

Let Assumptions 2 and 4 hold. Let $\delta < \delta^*$. Then for any $(\hat{x}, \hat{y}, \hat{i}) \in \mathbb{R}_0^{2n} \times S$,

$$\mathbb{E}\|\tilde{X}_{\hat{x},\hat{i}}(kh) - \tilde{X}_{\hat{y},\hat{i}}(kh)\|_h^p \leq C_1|\hat{x} - \hat{y}|^p e^{-\gamma_1 kh} \tag{2.4}$$

for all $k \in N_+$, where $\gamma_1 = 1/\bar{\theta} - \sigma\delta > 0$ and C_1 is positive constant independent of the initial data $(\hat{x}, \hat{y}, \hat{i})$.

Let $Z(t) = X_{\hat{x},\hat{i}}(t) - X_{\hat{y},\hat{i}}(t)$. For $(z, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}_+$, define a Lyapunov function

$$V_1(z, i, t) = \theta_i|z|^p \exp\left(\int_0^t \gamma(s)ds\right),$$

where $\gamma(t) = \frac{1}{\bar{\theta}} - \sigma(1 - \beta(t))$.

Problem 2

Lemma 7

Let Assumptions 2 and 4 hold. Let $\delta < \delta^*$. Then for any $(\hat{x}, \hat{i}) \in \mathbb{R}_n \times S$,

$$\mathbb{E}\|\tilde{X}_{\hat{x},\hat{i}}(kh)\|_h^p \leq C_2(1 + |\hat{x}|^p) \tag{2.5}$$

for all $k \in N_+$, where C_2 is a positive number independent of the initial data $(\hat{x}, \hat{i})$.

For $(x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}_+$, define a Lyapunov function

$$V_2(x, i, t) = \theta_i(1 + |x|^2)^{0.5p} \exp\left(\int_0^t \gamma(s)ds \right).$$

Problem 2

Theorem 8

Let Assumptions 2 and 4 hold. Let $\delta < \delta^*$. Then there exists a unique probability measure $\mu_h \in \mathcal{P}(\mathcal{C}_h)$ such that

$$\lim_{k\to\infty} d_\Phi(\mathcal{L}(\tilde{X}_{\hat{x},\hat{i}}(kh)), \mu_h) = 0$$

for all $(\hat{x}, \hat{i}) \in \mathbb{R}^n \times S$. In other words, the SDE (1.5) is asymptotically stable in distribution.

Problem 1

- The SDE (1.5) becomes (1.4) when $\delta = 0$ no mattter whatever value h is. In particular, $\tilde{X}_{\hat{x},\hat{i}}(t) \to X_{\hat{x},\hat{i}}(t)$ and $\mathcal{C}_h \to \mathbb{R}^n$ as $h \to 0$.
- Denote by $\mathcal{P}(\mathbb{R}^n)$ the family of probability measures on $\mathbb{R}^n$. For $P_1, P_2 \in \mathcal{P}(\mathbb{R}^n)$, define metric d_Ψ by

$$d_\Psi(P_1, P_2) = \sup_{\phi \in \Psi} \left| \int_{\mathbb{R}_n} \psi(\xi) P_1(dx) - \int_{\mathbb{R}^n} \phi(\xi) P_2(dx) \right|$$

where

$$\Psi = \{\psi : \mathbb{R}^n \to \mathbb{R} \text{ satisfying } |\psi(x) - \psi(y)| \leq |x - y| \text{ and } |\psi(x)| \leq 1 \text{ for } x, y \in \mathbb{R}^n\}.$$

- Denote by $\mathcal{L}(X_{\hat{x},\hat{i}}(t))$ the probability measure on $\mathbb{R}^n$ generated by $X_{\hat{x},\hat{i}}(t)$.

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- Denote by $\mathcal{L}(X_{\hat{x},\hat{i}}(t))$ the probability measure on $\mathbb{R}^n$ generated by $X_{\hat{x},\hat{i}}(t)$.

Problem 1

Theorem 9

Let Assumptions 2 and 4 hold. Then the SDE (1.4) is asymptotically stable in distribution.

Example 1

Let $B(t)$ be a scalar Brownian motion and $r(t)$ a Markov chain in $S = \{1, 2\}$ with generator

$$\Gamma = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$$

Consider a scalar hybrid ODE $\dot{x}(t) = f(x(t), r(t))$, where $f : \mathbb{R} \times S \to \mathbb{R}$ has the form

$$f(x, 1) = 1 + 2\cos(x) \text{ and } f(x, 2) = 2 + x.$$

Assume that the state is not observable in mode 1 so we could only apply a stochastic control in mode 2.

(i) Full time control. In terms of mathematics, the stochastically controlled SDE is

$$dX(t) = f(X(t), r(t))dt + u(X(t), r(t))dB(t), \tag{3.1}$$

where $u(x, 1) = 0$ and we will use $u(x, 2) = 6x$. It is easy to see that Assumption 2 is satisfied with

$$a_1 = 2,\ a_2 = 1,\ b_1 = c_1 = 0,\ b_2 = c_2 = 6.$$

Choose $p = 0.1$, then $\mathcal{A}$ defined by (2.1) becomes

$$\mathcal{A} = \begin{pmatrix} 2.8 & -3 \\ -1 & 2.61 \end{pmatrix},$$

which is a nonsingular M-matrix. In other words, Assumption 4 is satisfied too. By Theorem 9, the controlled SDE (3.1) is asymptotically stable in distribution.

(ii) Intermittent control. We will keep everything the same as in case (i) but use an intermittent control instead of the full time control. That is, the stochastically controlled system is now in the form of

$$dX(t) = f(X(t), r(t))dt + \beta(t)u(X(t), r(t))dB(t), \tag{3.2}$$

where $\beta(t)$ is the same as defined by (1.5). Noting

$$\mathcal{A}^{-1} = \begin{pmatrix} 0.6058496 & 0.6963788 \\ 0.2321263 & 0.6499536 \end{pmatrix},$$

by a straight computation, we get that $\theta_1 = \bar{\theta} = 1.3022284$, $\theta_2 = \underline{\theta} = 0.8820799$, $\sigma = 1.62$ and $\delta^* = 0.4740213$. By Theorem 8, we can hence conclude that if $\delta < 0.4740213$, the controlled SDE (3.2) is asymptotically stable in distribution.

Many Thanks!